Modern approaches to quantum gravity

Solution 1 Fall 2025

1. The variational principle of General Relativity

(a) This follows from $\delta(\sqrt{-g}g^{\mu\nu}R_{\mu\nu}) = \delta(\sqrt{-g})R + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}$ and $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

(b) This follows from some straightforward algebra after plugging $\delta\Gamma^{\lambda}_{\mu\nu}$ in $\delta R_{\mu\nu}$, or from arguing using normal coordinates.

(c) Expanding $g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon n_\mu n_\nu$, one obtains that the $n_\mu n_\nu$ terms cancel out and

$$\delta S_{\text{EH}} = \text{e.o.m.} + \frac{\epsilon}{16\pi G}\int d^{d-1}x\sqrt{\gamma}(n^\lambda\gamma^{\mu\nu}\nabla_\nu\delta g_{\mu\lambda} - n^\lambda\gamma^{\mu\nu}\nabla_\lambda\delta g_{\mu\nu}) \tag{1}$$

Using the Dirichlet boundary condition, the first term vanishes and we obtain the desired result.

(d) After straight-forward computation,

$$K_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu - \frac{\epsilon}{2}(n_\mu n^\rho\nabla_\rho n_\nu + n_\nu n^\rho\nabla_\rho n_\nu + n_\rho n_\nu\nabla_\mu n^\rho + n_\rho n_\mu\nabla_\nu n^\rho) \tag{2}$$

The last two terms vanish because $n_\rho\nabla_\alpha n^\rho = \frac{1}{2}\nabla_\alpha(n_\rho n^\rho) = 0$

(e) Using the hint,

$$\nabla_\mu n_\nu = (\nabla_\mu\alpha)\nabla_\nu f + \alpha\nabla_\mu\nabla_\nu f = \frac{1}{\alpha}n_\nu\nabla_\mu\alpha + \nabla_\mu\nabla_\nu f \tag{3}$$

Thus,

$$\nabla_\mu n_\nu - \nabla_\nu n_\mu = \frac{1}{\alpha}\left(n_\nu\nabla_\mu\alpha - n_\mu\nabla_\nu\alpha\right) \tag{4}$$

Also, using $n_\mu = \alpha\nabla_\mu f$,

$$\begin{aligned} n^\lambda n_\nu\nabla_\lambda n_\mu &= n^\lambda n_\nu(\nabla_\lambda\alpha)\nabla_\mu f + \alpha n^\lambda n_\nu\nabla_\lambda\nabla_\mu f \\ &= n^\lambda n_\nu(\nabla_\lambda\alpha)\nabla_\mu f + n^\lambda n_\nu\nabla_\mu(\underbrace{\alpha\nabla_\lambda f}_{n_\lambda}) - n^\lambda n_\nu(\nabla_\mu\alpha)\nabla_\lambda f \\ &= \frac{1}{\alpha}n_\mu n_\nu n^\lambda\nabla_\lambda\alpha - \frac{1}{\alpha}\epsilon n_\nu\nabla_\mu\alpha \end{aligned} \tag{5}$$

where in going from the first to the second line we interchange the λ and μ derivatives of the second term and introduced α in the first derivative, and from the second to the third we used $n^\lambda\nabla n_\lambda = 0$ and $n^\lambda\nabla_\lambda f = \frac{1}{\alpha}n^\lambda n_\lambda = \frac{1}{\alpha}\epsilon$. Thus,

$$\epsilon(n^\lambda n_\nu\nabla_\lambda n_\mu - n^\lambda n_\mu\nabla_\lambda n_\nu) = -\frac{1}{\alpha}n_\nu\nabla_\mu\alpha + \frac{1}{\alpha}n_\mu\nabla_\nu\alpha \tag{6}$$

since $\epsilon^2 = 1$. This is precisely the opposite of (4), giving the desired result.

(f) We have

$$\begin{aligned} K = g^{\mu\nu}K_{\mu\nu} &= \nabla^\mu n_\mu - \epsilon n^\lambda n^\mu \nabla_\lambda n_\mu \\ &= g^{\mu\nu}\nabla_\nu n_\mu - \epsilon n^\mu n^\nu \nabla_\nu n_\mu = \gamma^\nu{}_\mu \nabla_\nu n^\mu \end{aligned} \tag{7}$$

With Dirichlet boundary conditions, the only varying quantity is $\nabla \sim \partial + \Gamma$. The derivative piece ∂ does not vary under metric variations, thus

$$\delta K = \gamma^\nu{}_\mu \delta\Gamma^\mu_{\nu\rho} n^\rho = \frac{1}{2} n^\rho \gamma^{\nu\tau} \nabla_\rho \delta g_{\nu\tau} \tag{8}$$

where in the last equality we used $\delta\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\tau}(\nabla_\nu \delta g_{\rho\tau} + \nabla_\rho \delta g_{\nu\tau} - \nabla_\tau \delta g_{\nu\rho})$, the first and third contribution cancelling each other.

(g) To recap, we've shown in part (c) that,

$$\delta S_{\mathrm{EH}} = \text{e.o.m.} - \frac{\epsilon}{16\pi G} \int d^{d-1}x \sqrt{\gamma} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu} \tag{9}$$

and in part (f) that,

$$\delta K = \frac{1}{2} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu} \tag{10}$$

Thus, by defining

$$S_{\mathrm{GHY}} \equiv \frac{\epsilon}{8\pi G} \int d^{d-1}x \sqrt{\gamma} K \tag{11}$$

We have that with Dirichlet boundary conditions, $\delta(\sqrt{\gamma}) = 0$ and thus,

$$\delta S_{\mathrm{GHY}} = \frac{\epsilon}{16\pi G} \int d^{d-1}x \sqrt{\gamma} n^\lambda \gamma^{\mu\nu} \nabla_\lambda \delta g_{\mu\nu} \tag{12}$$

which cures the variational principle of GR,

$$\delta(S_{\mathrm{EH}} + S_{\mathrm{GHY}})\Big|_{\delta g_{\mu\nu}=0 \text{ on the boundary}} = \text{e.o.m.} \tag{13}$$

2. Spacetime energy

(a) On S_r^2 we will use capital indices we have the metric $\sigma_{AB} = \mathrm{diag}(r^2, r^2\sin^2(\theta))$. The normal vector at any point, living in the 3D space Σ_t is

$$\sigma^i = (1-\frac{2M}{r})^{\frac{1}{2}}(\partial_r)^i = (1-\frac{M}{r}+\mathcal{O}(r^{-2}))(\partial_r)^i \tag{14}$$

Here capital indices denote objects living on S_r^2, while normal latin indices denote objects living on Σ_t Therefore, the extrinsic curvature K is

$$K = \frac{1}{2}\sigma^{AB}\mathcal{L}_\sigma(\sigma_{AB}) = \frac{1}{2}\sigma^{AB}\sigma^r\partial_r\sigma_{AB} = \frac{2}{r}(1-\frac{M}{r}) \tag{15}$$

Subtracting the contribution from Minkowski space, we get $K - K^0 = -\frac{2M}{r^2}$, and we can compute the integral

$$E_{ADM} = -\frac{1}{8\pi}\lim_{r\to\infty}(4\pi r^2)(-\frac{2M}{r^2}) = M \tag{16}$$

The term containing the momentum π^{ij} can be neglected since N_i is exactly zero in these coordinates.

(b) As explained in the exercise statement, under an arbitrary transformation, the Lagrangian density $\mathcal{L}$ has to transform as

$$\delta\mathcal{L} = E\delta\phi + \partial_\mu(\Theta^\mu(\delta\phi)) \tag{17}$$

since $\int d^dx\delta\mathcal{L}$ should vanish on-shell (when $E = 0$). Moreover, under a generic symmetry δ_ϵ (it may be a local symmetry), the action needs to be invariant, meaning that the Lagrangian variation needs to be a total derivative, namely

$$\delta_\epsilon\mathcal{L} = \partial_\mu M^\mu(\epsilon) \tag{18}$$

Specializing (17) to a symmetry transformation $\delta = \delta_\epsilon$, we obtain the equation

$$E\delta_\epsilon\phi + \partial_\mu\Theta^\mu = \partial_\mu M^\mu \tag{19}$$

meaning that when $E = 0$, the equation can be written as

$$0 = \partial_\mu J^\mu \equiv \partial_\mu(\Theta^\mu - M^\mu) \tag{20}$$

where we identified the Noether current $J^\mu \equiv \Theta^\mu - M^\mu$.

(c) Now we will specify to Einstein-Hilbert gravity and compute the Noether current associated to diffeomorphisms δ_ξ. Under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$ we have,

$$\delta_\xi L = \xi^\mu\partial_\mu L\,, \qquad \delta_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \tag{21}$$

Thus,

$$\begin{aligned}\delta_\xi(\sqrt{g}L) &= \sqrt{g}\xi^\mu\partial_\mu L + \frac{1}{2}L\sqrt{g}g^{\mu\nu}\delta_\xi g_{\mu\nu}\\ &= \sqrt{g}\xi^\mu\partial_\mu L + \sqrt{g}L\nabla_\mu\xi^\mu\\ &= \sqrt{g}\xi^\mu\partial_\mu L + L\partial_\mu(\sqrt{g}\xi^\mu)\\ &= \partial_\mu(\sqrt{g}\xi^\mu L)\end{aligned} \tag{22}$$

We deduce that

$$M^\mu = \sqrt{g}\xi^\mu L = \frac{1}{16\pi G}\sqrt{g}R\xi^\mu \tag{23}$$

(d) Under arbitrary transformations δ we obtain, as in exercise 1,

$$\begin{aligned}
\delta S &= \text{e.o.m.} + \frac{1}{16\pi G}\int d^dx\sqrt{g}\nabla_\lambda(g^{\kappa\lambda}\nabla^\nu\delta g_{\kappa\nu} - g^{\kappa\nu}\nabla^\lambda\delta g_{\kappa\nu}) \\
&= \text{e.o.m.} + \int d^dx\partial_\lambda\left(\underbrace{\frac{1}{16\pi G}\sqrt{g}(g^{\kappa\lambda}\nabla^\nu\delta g_{\kappa\nu} - g^{\kappa\nu}\nabla^\lambda\delta g_{\kappa\nu})}_{=\Theta^\lambda}\right)
\end{aligned} \tag{24}$$

(e) First it is useful to rewrite Θ^μ using

$$\begin{aligned}
& g^{\mu\kappa}\nabla^\nu\delta g_{\kappa\nu} - g^{\kappa\nu}\nabla^\mu\delta g_{\kappa\nu} \\
&= \nabla^\nu\nabla^\mu\xi_\nu - \nabla^\nu\nabla_\nu\xi^\mu \\
&= g^{\alpha\beta}R^{\lambda}{}_{\beta}{}^{\mu}{}_{\alpha}\xi_\lambda = R^\mu{}_\nu\xi^\nu
\end{aligned} \tag{25}$$

where in going from the second to third line, we used the defining property of the Riemann tensor,

$$\nabla_\rho\nabla_\nu V_\mu - \nabla_\rho\nabla_\mu V_\nu \equiv R^\lambda{}_{\mu\nu\rho}V_\lambda.$$

We thus get, using (23), (24) and (25), that

$$J^\mu = \Theta^\mu - M^\mu = \frac{\sqrt{g}}{16\pi G}(R^\mu{}_\nu\xi^\nu - R\xi^\mu) \tag{26}$$

Using the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ to get rid of the R dependence, this can be written as

$$J^\mu = \frac{\sqrt{g}}{16\pi G}(-R^\mu{}_\nu\xi^\nu + 2G^\mu{}_\nu\xi^\nu) \tag{27}$$

Using (25), we can also write this as

$$J^\mu = \frac{\sqrt{g}}{16\pi G}\left[\nabla_\nu(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu) + 2G^{\mu\nu}\xi_\nu\right]. \tag{28}$$

as claimed.

(f) The Komar mass is

$$\begin{aligned}
E_k[\Sigma] =& \frac{1}{8\pi G}\int_\Sigma \sqrt{h}\; n^a\nabla_b(\nabla_a\xi^b - \nabla^b\xi_a) \\
=& \frac{1}{4\pi G}\int_{\partial\Sigma} \sqrt{\sigma}\; n^a\sigma^b\nabla_a\xi_b
\end{aligned} \tag{29}$$

where in the second line we used Stoke's theorem on Σ, and we used that ξ is an asymptotic Killing vector so that $\nabla^b\xi_a = -\nabla_a\xi^b$ at infinity. Note that the allowed diffeomorphisms are those who preserve the Dirichlet boundary conditions $\delta_\xi g_{\mu\nu} = 0$ on the boundary. This is why ξ^μ needs to be a Killing vector asymptotically.
On a curved spacetime, a timelike Killing vector generalises the usual concept of time translation generator. j^a represents the associated Noether current and E_k the associated conserved charge, i.e. energy.

(g) Using $n = \xi = \partial_t$, $\sigma = \partial_r$, the formula above gives

$$\begin{aligned} E_k =& \frac{1}{4\pi G}\int_{\partial\Sigma}\sqrt{\sigma}\; n^a\sigma^b\nabla_a\xi_b \\ =& \frac{1}{4\pi G}\lim_{r\to\infty}(4\pi r^2)(\Gamma^r_{tt}) = \lim_{r\to\infty}(r^2)\frac{M}{r^2} \\ =& \, M \end{aligned} \tag{30}$$